

The meromorphic non-integrability of the three-body problem

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Abstract

We study the planar three-body problem and prove the absence of a complete set of complex meromorphic first integrals in a neighborhood of the Lagrangian solution.

1. Introduction

The three-body problem is a mechanical system which consists of three mass points m_1, m_2, m_3 which attract each other according to the Newtonian law [16].

The practical importance of this problem arises from its applications to celestial mechanics: the bodies which constitute the solar system attract each other according to Newton's law, and the stability of this system on a long period of time is a fundamental question. Although Sundman [21] gave a power series solution to the three-body problem in 1913, it was not useful in determining the growth of the system for long intervals of time. Chazy [3] proposed in 1922 the first general classification of motion as $t \rightarrow \infty$. In view of the modern analysis [7], this stability problem leads to the problem of integrability of a Hamiltonian system i.e. the existence of a full set of analytic first integrals in involution. Poincaré [18] considered Hamiltonian functions $H(z, \mu)$ which in addition to z_1, \dots, z_{2n} also depended analytically on a parameter μ near $\mu = 0$. His theorem states that under certain assumptions about $H(z, 0)$, which are in general satisfied, the Hamiltonian system corresponding to $H(z, \mu)$ can have no integrals represented as convergent series in $2n+1$ variables z_1, \dots, z_{2n} and μ , other than the convergent series in H, μ . Based on this result he proved in 1889 the non-integrability of the restricted three-body problem [22]. However, this theorem does not assert anything about a fixed parameter value μ .

Bruns [2] showed in 1882 that the classical integrals are the only independent algebraic integrals of the problem of three bodies. His theorem has been extended by Painlevé [17], who has shown that every integral of the problem of n bodies which involves the velocities algebraically (whether the coordinates are involved algebraically or not) is a combination of the classical integrals.

However, citing [7] "One may agree with Winter [25] that these elegant negative results have no importance in dynamics, since they do not take into account the peculiarities of the behavior of phase trajectories. As far as first integrals are concerned, locally, in a neighborhood of a non-singular point, a complete set of independent integrals always exists. Whether they are algebraic or transcendent depends explicitly on the choice of independent variables. Therefore, the problem of the existence of integrals makes sense only when it is considered in the whole phase space or in a neighborhood of the invariant set ..."

Consider a complex-analytic symplectic manifold M , a holomorphic Hamiltonian vector field X_H on M and a non-equilibrium integral curve $\Gamma \subset M$. The nature of the relationship between the branching of solutions of a system of variational equations along Γ as functions of the complex time and the non-existence of first integrals of X_H goes back to the classical works of Kowalewskaya [6]. Ziglin [27] studied necessary conditions for an analytic Hamiltonian system with $n > 1$ degrees of freedom to possess n meromorphic independent first integrals in a sufficiently small neighborhood of the phase curve Γ . One can consider the monodromy group G of the normal variational equations along Γ . The key idea was that n independent meromorphic integrals of X_H must induce n independent rational invariants for G . Then, in order that Hamilton's equations have the above first integrals, it is necessary that for any two non-resonant transformations $g, g' \in G$ g must commute with g' . Although Ziglin formulated his result in terms of the monodromy group, it became quite recently [15,20] that much more could be achieved, under mild restrictions, by replacing this with the differential Galois group. Namely, one should check if its identity component, under Zariski's topology, is abelian.

The collinear three-body problem was proved to be non-integrable near triple collisions by Yoshida [26] based on Ziglin's analysis.

The present paper is devoted to the non-integrability of the planar three-body problem.

In 1772 Lagrange [8] discovered the particular solution in which three bodies form an equilateral triangle and each body describes a conic.

Moeckel [14] has shown that for a small angular momentum there exist orbits homoclinic to the Lagrangian elliptical orbits and heteroclinic between them. Consequently in this case the problem is not-integrable. Nevertheless, it was observed that for a large angular momentum and for certain masses of two bodies which are relatively small compared to the third one, the circular Lagrangian orbits are stable and, a priori, the system can be integrable near these solutions. Topan [23] found some examples of such transcendental integrals in certain configurations of the restricted three-body problem.

Our approach consists of applying the methods related to [27,15] to the Lagrangian parabolic orbits. This means that we will study the integrability of the problem in a sufficiently small complex neighborhood of these solutions.

The plan of the paper is follows. In Section 2, following Whittaker, we introduce the reductions of the planar three-body problem from the Hamiltonian system of 6 degrees of freedom to 3 degrees of freedom. Section 3 is devoted to a parametrization of the Lagrangian parabolic solution. Section 4 contains the normal variational equations along this solution. In Section 5 we study the monodromy group of these equations. In Section 6, applying the Ziglin's method, we prove that for the three-body problem there are no two additional meromorphic first integrals in a connected neighborhood of the Lagrangian parabolic solution (Theorems 6.2-6.3). Section 7 contains a dynamical interpretation of above theorems in connection with a theory of splitting and transverse intersection of asymptotic manifolds.

2. The reduction of the problem

Following Whittaker [24] let (x_1, x_2) be the coordinates of m_1 , (x_3, x_4) the coordinates of m_2 , and (x_5, x_6) the coordinates of m_3 . Let $y_r = m_k \frac{dx_r}{dt}$, where k denotes the greatest integer in $\frac{1}{2}(r+1)$. The equations of motion are

$$(2.1) \quad \frac{dx_r}{dt} = \frac{\partial H_1}{\partial y_r}, \quad \frac{dy_r}{dt} = -\frac{\partial H_1}{\partial x_r}, \quad (r = 1, 2, \dots, 6),$$

where

$$H_1 = \frac{1}{2m_1}(y_1^2 + y_2^2) + \frac{1}{2m_2}(y_3^2 + y_4^2) + \frac{1}{2m_3}(y_5^2 + y_6^2) - m_3m_2\{(x_3 - x_5)^2 + (x_4 - x_6)^2\}^{-1/2} - m_3m_1\{(x_5 - x_1)^2 + (x_6 - x_2)^2\}^{-1/2} - m_1m_2\{(x_1 - x_3)^2 + (x_2 - x_4)^2\}^{-1/2}.$$

This is a Hamiltonian system with 6 degrees of freedom which admits 4 first integrals:

$T_1 = H_1$ - the energy,

$T_2 = y_1 + y_3 + y_5, T_3 = y_2 + y_4 + y_6$ - the components of the impulse of the system,

$T_4 = y_1x_2 + y_3x_4 + y_5x_6 - x_1y_2 - x_3y_4 - x_5y_6$ - the integral of angular momentum of the system.

The system (2.1) can be transformed to a system with 4 degrees of freedom by the following canonical change (Poincaré, 1896)

$$x_r = \frac{\partial W_1}{\partial y_r}, \quad g_r = \frac{\partial W_1}{\partial l_r}, \quad (r = 1, 2, \dots, 6),$$

where

$$(2.2) \quad W_1 = y_1l_1 + y_2l_2 + y_3l_3 + y_4l_4 + (y_1 + y_3 + y_5)l_5 + (y_2 + y_4 + y_6)l_6.$$

Here (l_1, l_2) are the coordinates of m_1 relative to axes through m_3 parallel to the fixed axes, (l_3, l_4) are the coordinates of m_2 relative to the same axes, (l_5, l_6) are the coordinates of m_3 relative to the original axes, (g_1, g_2) are the components of impulse of m_1 , (g_3, g_4) are the components of impulse of m_2 , and (g_5, g_6) are the components of impulse of the system. It can be shown that in the system of the center of masses the corresponding equations for l_5, l_6, g_5, g_6 disappear from the system and the reduced system takes the following form

$$(2.3) \quad \frac{dl_r}{dt} = \frac{\partial H_2}{\partial g_r}, \quad \frac{dg_r}{dt} = -\frac{\partial H_2}{\partial l_r}, \quad (r = 1, 2, 3, 4),$$

with the Hamiltonian

$$H_2 = \frac{M_1}{2}(g_1^2 + g_2^2) + \frac{M_2}{2}(g_3^2 + g_4^2) + \frac{1}{m_3}(g_1g_3 + g_2g_4) - \frac{m_3m_2}{\rho_1} - \frac{m_1m_3}{\rho_2} + \frac{m_1m_2}{\rho_3},$$

where

$$\rho_1 = \sqrt{l_3^2 + l_4^2}, \quad \rho_2 = \sqrt{l_1^2 + l_2^2}, \quad \rho_3 = \sqrt{(l_1 - l_3)^2 + (l_2 - l_4)^2},$$

are the mutual distances of the bodies and $M_1 = m_1^{-1} + m_3^{-1}$, $M_2 = m_2^{-1} + m_3^{-1}$.

This system admits two first integrals in involution

$K_1 = H_2$ – the energy,

$K_2 = g_2l_1 + g_4l_3 + g_6l_5 - g_1l_2 - g_3l_4 - g_5l_6 = k$ – the integral of angular momentum.

Let us suppose that the Hamiltonian system (2.3) possesses a first integral K different from $K_{1,2}$.

Definition 2.1 The first integral K of the system (2.3) is called *meromorphic* if it is representable as a ratio

$$K = \frac{R(l, g)}{Q(l, g)},$$

where R, Q are analytic functions of the variables l_i, g_i , $1 \leq i \leq 4$.

It can be shown [24] that the system (2.3) possesses an ignorable coordinate which will make possible a further reduction.

Let us make the following canonical transformation

$$(2.4) \quad l_r = \frac{\partial W_2}{\partial g_r}, \quad p_r = \frac{\partial W_2}{\partial q_r}, \quad (r = 1, 2, 3, 4),$$

where

$$W_2 = g_1q_1\cos q_4 + g_2q_1\sin q_4 + g_3(q_2\cos q_4 - q_3\sin q_4) + g_4(q_2\sin q_4 + q_3\cos q_4).$$

Here q_1 is the distance m_3m_1 ; q_2 and q_3 are the projections of m_2m_3 on, and perpendicular to m_1m_3 ; p_1 is the component of momentum of m_1 along m_3m_1 ; p_2 and p_3 are the components of momentum of m_2 parallel and perpendicular to m_3m_1 .

One can write the new equations as follows

$$(2.5) \quad \frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, 3),$$

and

$$(2.5.a) \quad \frac{dq_4}{dt} = \frac{\partial H}{\partial p_4}, \quad \frac{dp_4}{dt} = 0,$$

with the Hamiltonian

$$H = \frac{M_1}{2} \left\{ p_1^2 + \frac{1}{q_1^2} P^2 \right\} + \frac{M_2}{2} (p_2^2 + p_3^2) + \frac{1}{m_3} \left\{ p_1p_2 - \frac{p_3}{q_1} P \right\} - \frac{m_1m_3}{r_1} - \frac{m_3m_2}{r_2} - \frac{m_1m_2}{r_3},$$

$$P = p_3q_2 - p_2q_3 - p_4,$$

where

$$r_1 = q_1, \quad r_2 = \sqrt{q_2^2 + q_3^2}, \quad r_3 = \sqrt{(q_1 - q_2)^2 + q_3^2},$$

are the mutual distances of the bodies.

Since $p_4 = k = \text{const}$ the system (2.5) is a closed Hamiltonian system with 3 degrees of freedom. If this system is integrated then q_4 can be found by a quadrature from (2.5.a).

Proposition 2.2 *If the Hamiltonian system (2.3) admits the full set of functionally independent meromorphic first integrals in involution $\{K_1, K_2, K_3, K_4\}$ then the system (2.5) possesses two functionally independent additional first integrals $\{H_1, H_2\}$ which are meromorphic functions of the variables q_i, p_i , $1 \leq i \leq 3$.*

This is the obvious consequence of the canonical change (2.4).

3. A parametrization of the parabolic Lagrangian solution

The equations (2.1) admit an exact solution discovered by Lagrange [8] in which the triangle formed by the three bodies is equilateral and the trajectories of the bodies are similar conics with one focus at the common barycenter. For the reduced form (2.5) the equality of the mutual distances gives

$$(3.1) \quad q_1 = q, \quad q_2 = \frac{q}{2}, \quad q_3 = \frac{\sqrt{3}q}{2},$$

where $q = q(t)$ is an unknown function. Substituting (3.1) into (2.5) one can show that

$$(3.2) \quad p_1 = p, \quad p_2 = Ap + \frac{B}{q}, \quad p_3 = Cp + \frac{D}{q},$$

with $p = p(t)$ unknown and A, B, C, D are the following constants

$$A = \frac{m_2(m_3 - m_1)}{m_1 S_3}, \quad B = -\frac{\sqrt{3}k S_1 m_2 m_3}{S_2 S_3}, \quad C = \frac{\sqrt{3}m_2(m_1 + m_3)}{m_1 S_3},$$

$$D = -\frac{k m_2 (S_2 + m_1 m_2 - m_3^2)}{S_2 S_3},$$

where

$$S_1 = m_1 + m_2 + m_3, \quad S_2 = m_1 m_2 + m_2 m_3 + m_3 m_1, \quad S_3 = m_2 + 2m_3.$$

Substituting (3.1), (3.2) into the integral of energy $H = h = \text{const}$ we obtain the following relation between q and p

$$(3.3) \quad ap^2 + \frac{bp}{q} + \frac{c}{q} + \frac{d}{q^2} = h,$$

where

$$a = \frac{2S_1 S_2}{m_1^2 S_3^2}, \quad b = -\frac{2\sqrt{3}k m_2 S_1}{m_1 S_3^2}, \quad c = -S_2, \quad d = \frac{2k^2 S_1 (m_2^2 + m_2 m_3 + m_3^2)}{S_3^2 S_2}.$$

Moreover, from (2.5) we have

$$(3.4) \quad \frac{dq}{dt} = \left(M_1 + \frac{A}{m_3} \right) p + \frac{B}{m_3 q}$$

The equations (3.1), (3.2), (3.3), (3.4) define all Lagrangian particular solutions and contain two free parameters: k and h .

Consider the case of zero energy $h = 0$ and $k \neq 0$. Then there exists a parabolic particular solution in the sense that the limit velocity goes to zero when the bodies approach infinity and each body describes a parabola.

Putting $w = pq$ one can find by using of (3.3) q, p as the functions of w

$$(3.5) \quad q = P(w), \quad p = \frac{w}{P(w)},$$

where $P(w) = -(aw^2 + bw + d)/c$.

Let $M = \mathbb{C}^6$ be the complexified phase space of the system (2.5). Then (3.5), (3.1), (3.2) define a parametrized parabolic integral curve $\Gamma \in M$ with the parameter $w \in \mathbb{CP}^1$.

4. The normal variational equations

Let $z = (q_1, q_2, q_3, p_1, p_2, p_3)$, $z \in M$. One can obtain the variational equations of the system (2.5) along the integral curve Γ

$$(4.1) \quad \frac{d\zeta}{dt} = JH_{zz}(\Gamma)\zeta, \quad \zeta \in T_\Gamma M,$$

where H_{zz} is the Hessian matrix of Hamiltonian H at Γ and J is the 6×6 matrix

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where E is the identity 3×3 matrix .

These equations admit the linear first integral $F = (\zeta, H_z(\Gamma))$, where $H_z = \text{grad}(H)$ and can be reduced on the normal 5-dimensional bundle $G = T_\Gamma M / T\Gamma$ of Γ . After the restriction of (4.1) on the surface $F = 0$ we obtain *normal variational equations* (NVE) [27] which are the system of 4 equations

$$(4.2) \quad \frac{d\eta}{dt} = \tilde{A}(\Gamma)\eta, \quad \eta \in \mathbb{C}^4,$$

where \tilde{A} is a 4×4 matrix depending on Γ .

We can obtain NVE in the following natural way applying Whittaker's procedure [24] of reducing the order of the Hamiltonian system (2.5).

Fixing the level of energy $h = 0$ one can find p_1 as a function of the other variables from the equation $H(q, p) = 0$ which takes the following form

$$a_1 p_1^2 + b_1 p_1 + c_1 = 0,$$

where a_1, b_1, c_1 are known functions depending on p_2, p_3, q_1, q_2, q_3 .

Solving this equation we get two solutions for p_1

$$p_1 = \frac{-b_1 + \sqrt{\Delta}}{2a_1} = K_+ \quad \text{and} \quad p_1 = \frac{-b_1 - \sqrt{\Delta}}{2a_1} = K_-,$$

where $\Delta = b_1^2 - 4a_1 c_1$.

By substituting the Lagrangian solution given by (3.1), (3.2), (3.5) in these relations we choose the root $p_1 = K_-$ as corresponding to this solution.

The functions $q_r(t), p_r(t)$, $r = 2, 3$ satisfy the canonical equations

$$(4.3) \quad \frac{dq_r}{dq_1} = \frac{\partial K}{\partial p_r}, \quad \frac{dp_r}{dq_1} = -\frac{\partial K}{\partial q_r}, \quad (r = 2, 3),$$

where $K = -K_-$ and q_1 is taken as the new time.

The system (4.3) is a nonautonomous Hamiltonian system with 2 degrees of freedom which has the same integral curve Γ . Notice that K is not more a first integral.

It is useful to pass now to the new time $(q_1 = q) \rightarrow w$. From the formulas (3.3), (3.5) we have

$$(4.4) \quad q = \frac{aw^2 + bw + d}{c}, \quad dq = -\frac{2aw + b}{c}dw.$$

The resulting NVE (4.2) are obtained as the variational equations of the system (4.3) near the integral curve Γ and after the substitution (4.4) take the form

$$(4.5) \quad \frac{d\eta}{dw} = \tilde{A}(\Gamma)\eta, \quad \eta \in \mathbb{C}^4,$$

where \tilde{A} is a 4×4 matrix whose elements are rational functions of w .

We can represent \tilde{A} in the following block form

$$\tilde{A} = \begin{pmatrix} M_3^T & M_2 \\ -M_1 & -M_3 \end{pmatrix},$$

where M_1, M_2, M_3 are 2×2 matrices and M_3^T means the transposition of M_3 .

The matrix M_1 is symmetric and has the following form

$$M_1 = \frac{1}{S_1^3 L^2 Z^2} \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix},$$

where $L(w)$ is the linear polynomial

$$L = l_1 w + l_2,$$

and $l_1 = 2S_2$, $l_2 = -\sqrt{3}m_1 m_2 k$.

$Z(w)$ is the following quadratic polynomial

$$Z = z_1 w^2 + z_2 w + z_3,$$

where $z_1 = S_2^2$, $z_2 = -\sqrt{3}m_1 m_2 k S_2$, $z_3 = k^2 m_1^2 (m_2^2 + m_2 m_3 + m_3^2)$.

The coefficients n_{ij} have the form

$$n_{11} = A_1 w^2 + A_2 w + A_3, \quad n_{12} = A_4 w^2 + A_5 w + A_6, \quad n_{22} = A_7 w^2 + A_8 w + A_9,$$

where A_i are constants depending on the masses m_1 , m_2 , m_3 and k .

The matrix M_2 has the following expression

$$M_2 = \frac{4S_1 Z}{S_2 S_3^3 m_1^4 m_2 m_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the matrix M_3 we have

$$M_3 = \frac{1}{m_1 S_1 L Z} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

where

$$\begin{aligned} m_{11} &= B_1 w^2 + B_2 w + B_3, & m_{12} &= B_4 w^2 + B_5 w + B_6, & m_{21} &= B_7 w^2 + B_8 w + B_9, \\ m_{22} &= B_{10} w^2 + B_{11} w + B_{12}, \end{aligned}$$

and B_j are constants depending on m_1 , m_2 , m_3 and k .

The system (4.5) has four singular points w_1 , w_2 , w_3 , w_4 in the complex plane:

$$w_1 = \infty,$$

– the infinity.

$$w_2 = \frac{\sqrt{3}m_1 m_2 k}{2S_2},$$

– the root of $L = 0$.

$$(4.6) \quad w_3 = \frac{(\sqrt{3}m_2 + iS_3)km_1}{2S_2}, \quad w_4 = \frac{(\sqrt{3}m_2 - iS_3)km_1}{2S_2},$$

– the corresponding roots of the quadratic equation $Z = 0$ where $i^2 = -1$.

Notice that the expressions for $w_{2,3,4}$ have a rational form on the masses.

The singularities w_i , $1 \leq i \leq 4$ have a clear mechanical sense: w_1 corresponds to the motion of the bodies at infinity, w_2 defines the moment of the maximal approach.

It is easy to see from (4.6) that if the angular momentum constant $k = 0$, then $w_2 = w_3 = w_4 = 0$ and we have a triple collision of the bodies at the moment of time $w = 0$. If $k \neq 0$ then by the lemma of Sundman there are no triple collisions in the real phase space and $w_{3,4}$ become complex.

Since the expression for p given in (3.5) becomes infinity when $w \rightarrow w_{3,4}$, formally, we can consider w_3 and w_4 as corresponding to the “complex” collisions which tend to $w = 0$ as $k \rightarrow 0$.

It was noted by Schaeffe [19] that the equations (4.5) can be reduced to fuchsian form.

In order to do it, consider the linear change of variables

$$(4.7) \quad \eta = Cx,$$

where $\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$, $x = (x_1, x_2, x_3, x_4)^T$ and $C = \text{diag}(LZ, LZ, 1, 1)$.

In new variables the system (4.5) takes the following form

$$(4.8) \quad \frac{dx}{dw} = \left(\frac{A(k)}{w - w_2} + \frac{B(k)}{w - w_3} + \frac{C(k)}{w - w_4} \right) x, \quad x \in \mathbb{C}^4,$$

where $A(k), B(k), C(k)$ are known constant 4×4 matrices depending on m_1, m_2, m_3 and k .

Under the assumption $k \neq 0$ we can exclude the parameter k from the system (4.8) by using the change of time $w = kt$. As a result, one obtains

$$(4.9) \quad \frac{dx}{dt} = \left(\frac{A}{t - t_0} + \frac{B}{t - t_1} + \frac{C}{t - t_2} \right) x,$$

where

$$t_0 = \frac{\sqrt{3}m_1m_2}{2S_2}, \quad t_1 = \frac{m_1(\sqrt{3}m_2 + iS_3)}{2S_2}, \quad t_2 = \frac{m_1(\sqrt{3}m_2 - iS_3)}{2S_2}.$$

and

$$A = \frac{\tilde{M}(t_0)}{(t_0 - t_1)(t_0 - t_2)}, \quad B = \frac{\tilde{M}(t_1)}{(t_1 - t_2)(t_1 - t_0)}, \quad C = \frac{\tilde{M}(t_2)}{(t_2 - t_1)(t_2 - t_0)}.$$

Here, $\tilde{M}(w)$ is the following matrix

$$\tilde{M}(w) = \begin{pmatrix} L Z M_3^T - \frac{\partial L Z}{\partial w} E & M_2 \\ -L^2 Z^2 M_1 & -L Z M_3 \end{pmatrix},$$

where one should put $k = 1$.

The system (4.9) is defined on a connected Riemann surface $X = \mathbb{CP}^1 / \{t_0, t_1, t_2, \infty\}$.

It turns out that the matrix A is real and the matrices $B = R + iJ$, $C = R - iJ$ are complex conjugate being R and J real matrices. It will simplify matters further if we choose the units of masses as follows

$$m_1 = \alpha, \quad m_2 = \beta, \quad m_3 = 1, \quad 0 < \alpha \leq \beta \leq 1.$$

In Appendix A we write the expressions for A, R, J with help of MAPLE.

5. The monodromy group of the system (4.9)

Let $\Sigma(t)$ be a solution of the matrix equation (4.9)

$$(5.1) \quad \frac{d}{dt}\Sigma = \left(\frac{A}{t - t_0} + \frac{B}{t - t_1} + \frac{C}{t - t_2} \right) \Sigma,$$

with the initial condition $\Sigma(\tau) = I$, $\tau \in X$ where I is the unit 4×4 matrix.

It can be continued along a closed path γ with end points at τ . We obtain the function $\tilde{\Sigma}(t)$ which also satisfies (5.1). From linearity of (5.1) it follows that there exists a complex 4×4 matrix T_γ such that $\tilde{\Sigma}(t) = \Sigma(t)T_\gamma$. The set of matrices $G = \{T_\gamma\}$ corresponding to all closed curves in X is a group. This group is called the *monodromy group* of the linear system (4.9). Let T_i be the elements of G corresponding to circuits around the singular points $t = t_i$, $i = 0, 1, 2$. Then the monodromy group G is formed by T_0, T_1, T_2 . Denote by $T_\infty \in G$ the element corresponding to a circuit around the point $t = \infty$.

Lemma 5.1 *The following assertions about the monodromy group G hold*

a) $T_0 = I$ – is the unit matrix and

$$(5.2) \quad T_1 T_2 = T_\infty^{-1}.$$

b) There exist two non-singular matrices U, V such that

$$U^{-1}T_1U = V^{-1}T_2V = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

c) The matrix T_∞ has the following eigenvalues

$$(5.3) \quad \text{Spectr}(T_\infty) = \{e^{2\pi i\lambda_1}, e^{2\pi i\lambda_2}, e^{-2\pi i\lambda_1}, e^{-2\pi i\lambda_2}\},$$

where

$$(5.4) \quad \lambda_1 = \frac{3}{2} + \frac{1}{2}\sqrt{13 + \sqrt{\theta}}, \quad \lambda_2 = \frac{3}{2} + \frac{1}{2}\sqrt{13 - \sqrt{\theta}},$$

and

$$\theta = 144 \left(1 - 3\frac{S_2}{S_1^2}\right), \quad S_1 = \alpha + \beta + 1, \quad S_2 = \alpha\beta + \alpha + \beta.$$

Moreover,

$$\text{Spectr}(T_\infty) \neq \{1, 1, 1, 1\}.$$

Proof. a) The matrix A has the eigenvalues $\{-1, -1, 0, 0\}$. Following the general theory of the linear differential equations let us write the general solution of the system (4.9) near the singular point $t = t_0$ as follows

$$x(t) = c_1X_1(t) + c_2X_2(t) + c_3X_3(t) + c_4X_4(t),$$

where $c_1, \dots, c_4 \in \mathbb{C}$ are arbitrary constants and

$$(5.5) \quad \begin{aligned} X_1(t) &= \frac{a_{-1}}{t-t_0} + a_0 + a_1(t-t_0) + \dots, & X_2(t) &= \frac{b_{-1}}{t-t_0} + b_0 + b_1(t-t_0) + \dots, \\ X_3(t) &= c_0 + c_1(t-t_0) + \dots, & X_4(t) &= d_0 + d_1(t-t_0) + \dots, \end{aligned}$$

where $a_i, b_i, c_i, d_i \in \mathbb{C}^4$ are some constant vectors.

By substituting (5.5) in (4.9) one can find a_i, b_i, c_i, d_i and show that the vectors $X_1(t), X_2(t), X_3(t), X_4(t)$ are functionally independent and meromorphic in a small neighborhood of the point $t = t_0$. This implies that the element T_0 of the monodromy group G corresponding to a circuit around t_0 is the unit matrix. Obviously we should have $T_0T_1T_2 = T_\infty^{-1}$. From this fact the relation (5.2) follows.

b) The matrices B, C have the same eigenvalues $\{-2, -1, 0, 1\}$. It can be shown by a straightforward calculation that near the singular point $t = t_1$ the general solution of the system (4.9) can be represented as

$$x(t) = c_1Y_1(t) + c_2Y_2(t) + c_3Y_3(t) + c_4Y_4(t),$$

where $c_1, \dots, c_4 \in \mathbb{C}$ are arbitrary constants and

$$\begin{aligned} Y_1(t) &= e_1(t-t_1) + e_2(t-t_1)^2 + \dots, & Y_2 &= f_0 + f_1(t-t_1) + \dots + C_1 \ln(t-t_1)Y_1(t), \\ Y_3(t) &= \frac{g_{-1}}{t-t_1} + g_0 + g_1(t-t_1) + \dots, \\ Y_4(t) &= \frac{h_{-2}}{(t-t_1)^2} + \frac{h_{-1}}{t-t_1} + \dots + C_2 \ln(t-t_1)(f_0 + f_1(t-t_1) + \dots) + C_3 \ln(t-t_1)Y_1(t), \end{aligned}$$

where $e_i, f_i, g_i, h_i \in \mathbb{C}^4$ are some constant vectors and C_1, C_2, C_3 are parameters depending on the masses α, β .

For C_1, C_2 one can find

$$C_1 = \frac{9\beta\alpha^3(\beta+2)^2(\alpha\beta+\alpha+\beta)}{4(\alpha+\beta+1)^3}, \quad C_2 = iC_1.$$

The matrix $\Sigma(t) = (Y_1, Y_2, Y_3, Y_4)$ represents the solution of the system (5.1) in a small neighborhood of the point $t = t_1$. After going around of t_1 we get $\tilde{\Sigma}(t) = \Sigma(t)M$ where

$$M = \begin{pmatrix} 1 & 2\pi i C_1 & 0 & 2\pi i C_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\pi i C_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $C_1 \neq 0, C_2 \neq 0$ for $\alpha > 0, \beta > 0$, there exists a non-singular matrix T such that

$$(5.6) \quad T^{-1}MT = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is the Jordan form of M .

The matrix T_1 is similar to M and therefore has the same Jordan form (5.6). Repeating the analogous arguments for the matrix T_2 we deduce that the same assertion holds for the monodromy matrix T_2 . Notice that the existence of logarithmic branching near some Lagrangian solutions in three body problem was first observed by H. Block (1909) and J.F. Chazy (1918) (see for instance [1]).

c) Consider the matrix $A_\infty = -(A + B + C)$. Then there exists (see for example [4]) a non-singular matrix W such that

$$(5.7) \quad T_\infty = W^{-1}e^{2\pi i A_\infty}W.$$

Appendix A contains the expressions for the elements of the matrix A_∞ . One can calculate its eigenvalues

$$\text{Spectr}(A_\infty) = \{\lambda_1, \lambda_2, 3 - \lambda_1, 3 - \lambda_2\},$$

where $\lambda_{1,2}$ are given in (5.3).

One can easy check that

$$(5.8) \quad 0 \leq \sqrt{\theta} < 12,$$

for all $\alpha > 0, \beta > 0$.

With the help of (5.7) we obtain for the eigenvalues of the matrix T_∞ the expression (5.3).

Let us suppose now that $\text{Spectr}(T_\infty) = \{1, 1, 1, 1\}$. Then according to (5.4) we obtain

$$(5.9) \quad \sqrt{13 + \sqrt{\theta}} = n_1, \quad \sqrt{13 - \sqrt{\theta}} = n_2, \quad n_1, n_2 \in \mathbb{Z}.$$

Hence, in view of (5.8), the number $r = \sqrt{\theta}$ is an integer $0 \leq r \leq 11$. The simple calculation shows that for these r the relations (5.9) are not fulfilled. This implies that

$$\text{Spectr}(T_\infty) \neq \{1, 1, 1, 1\}.$$

The proof of Lemma 5.1 is completed. \square

6. Nonexistence of additional meromorphic first integrals

We call the planar three-body problem (2.1) *meromorphically* integrable near the Lagrangian parabolic solution Γ , defined in Section 3, if the corresponding Hamiltonian system (2.3) possesses a complete set of complex meromorphic first integrals (see Definition 2.1) in involution in a connected neighborhood of Γ . Recall that equations (2.3) describe the motion of bodies in the system of the center of masses.

From Proposition 2.2 it follows that in this case the system (2.5) admits two additional first integrals which are meromorphic and functionally independent in the same neighborhood.

Theorem 6.1 *For $k \neq 0$ for the Hamiltonian system (2.5) there are no two functionally independent additional first integrals, meromorphic in a connected neighborhood of the Lagrangian parabolic solution Γ .*

Proof. Suppose that the Hamiltonian system (2.5) admits two functionally independent first integrals H_1, H_2 , meromorphic in a connected neighborhood of the Lagrangian parabolic solution Γ and functionally independent together with H . According to Ziglin [27] in this case the NVE (4.5) have two functionally independent meromorphic integrals F_1, F_2 which are single-valued in a complex neighborhood of the Riemann surface $\Gamma = \mathbb{CP}^1/\{t_0, t_1, t_2, \infty\}$. The linear system (4.9) was obtained from (4.5) by the linear change of variables (4.7) and the change of the time $w = kt, k \neq 0$. Therefore, it possesses two functionally independent meromorphic integrals I_1, I_2 . From this fact the following lemma is deduced

Lemma 6.2 (Ziglin [27]) *The monodromy group G of the system (4.9) has two rational, functionally independent invariants J_1, J_2 .*

In appropriate coordinates, according to b) of Lemma 5.1, the monodromy transformation T_1 can be written as follows

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I + D,$$

where I is the unit matrix and

$$(6.1) \quad D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the monodromy matrix T_2 one writes

$$T_2 = I + R,$$

where

$$(6.2) \quad R = \tilde{V} D \tilde{V}^{-1} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix},$$

with some unknowns $a_i, b_i, c_i, d_i \in \mathbb{C}$ and a nonsingular matrix \tilde{V} .

Let us input the following linear differential operators

$$\delta = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3},$$

and

$$\Delta = \left(\sum_{i=1}^4 a_i x_i \right) \frac{\partial}{\partial x_1} + \left(\sum_{i=1}^4 b_i x_i \right) \frac{\partial}{\partial x_2} + \left(\sum_{i=1}^4 c_i x_i \right) \frac{\partial}{\partial x_3} + \left(\sum_{i=1}^4 d_i x_i \right) \frac{\partial}{\partial x_4}.$$

Lemma 6.3 *Let J be a rational invariant of the monodromy group G , then the following relations hold*

$$\delta J = 0, \quad \Delta J = 0.$$

Proof. For an arbitrary $n \in \mathbb{N}$ we have $T_1^n = I + nD$, hence $J(T_1^n x) = J(x + nDx)$. Expanding the last expression in Taylor series we obtain

$$(6.3) \quad J(T_1^n x) = J(x) + n\delta J(x) + \sum_{i=2}^{\infty} n^i r_i(x),$$

where $r_i(x)$ are some rational functions.

In view of $J(T_1^n x) = J(x)$ and the fact that $J(x)$ is a rational function on x , the second term of (6.3) gives $\delta J = 0$. The relation $\Delta J = 0$ is deduced by analogy from the identity $J(T_2 x) = J(x)$. \square

Case (1). Assume that invariants J_1, J_2 depend on x_2, x_4 only. By Lemma 6.3 we have

$$(6.4) \quad \Delta J_1 = 0, \quad \Delta J_2 = 0.$$

It can be verified that the equations (6.4) imply the conditions $b_i = 0, d_i = 0, 1 \leq i \leq 4$. Accordingly, the matrix R may be written

$$(6.5) \quad R = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

One can find the characteristic polynomial $P(\lambda) = \det(R - \lambda I)$ of R

$$(6.6) \quad P(\lambda) = \lambda^4 - (a_1 + c_3)\lambda^3 + (a_1 c_3 - c_1 a_3)\lambda^2.$$

In view of (6.1), (6.2) all eigenvalues of the matrix R are equal to 0, thus, with help of (6.6) we get

$$(6.7) \quad a_1 + c_3 = 0, \quad a_1 c_3 = c_1 a_3.$$

The matrix $T_1 T_2$ takes the following form

$$T_1 T_2 = \begin{pmatrix} a_1 + 1 & a_2 + 1 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 + 1 & c_4 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\text{Spectr}(T_1 T_2) = \{1, 1, s + f, s - f\},$$

where

$$(6.8) \quad s = 1 + \frac{a_1 + c_3}{2}, \quad f = \frac{\sqrt{a_1^2 + c_3^2 + 4c_1 a_3 - 2a_1 c_3}}{2}.$$

The straightforward calculation by using (6.7) and (6.8) shows that the eigenvalues of the matrix $T_1 T_2$ are equal to $\{1, 1, 1, 1\}$. According to (5.2) these must be the eigenvalues of the matrix T_∞ which is in contradiction to c) of Lemma 5.1.

Case(2). Assume that even one from the invariants J_1, J_2 depends on x_1 or x_3 . Let, for example

$$(6.9) \quad \frac{\partial J_1}{\partial x_1} \neq 0.$$

It is useful to consider two additional linear operators $\delta_1 = [\delta, \Delta]$ and $\delta_2 = -\frac{1}{2}[\delta, \delta_1]$.

One has

$$\delta_1 = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3} + f_4 \frac{\partial}{\partial x_4}, \quad \delta_2 = (b_1 x_2 + b_3 x_4) \frac{\partial}{\partial x_1} + (d_1 x_2 + d_3 x_4) \frac{\partial}{\partial x_3},$$

where

$$(6.10) \quad \begin{aligned} f_1 &= -b_1 x_1 + (a_1 - b_2) x_2 - b_3 x_3 + (a_3 - b_4) x_4, & f_2 &= b_1 x_2 + b_3 x_4, \\ f_3 &= -d_1 x_1 + (c_1 - d_2) x_2 - d_3 x_3 + (c_3 - d_4) x_4, & f_4 &= d_1 x_2 + d_3 x_4. \end{aligned} \quad ,$$

We deduce from $\delta J_i = \Delta J_i = 0$ that

$$\delta_1 J_i = 0, \quad \delta_2 J_i = 0, \quad i = 1, 2.$$

Consider the partial differential equation $\delta J = 0$. Solving it one finds that $J = K(Y_1, Y_2, Y_3)$ where $K(y_1, y_2, y_3)$ is an arbitrary function and

$$(6.11) \quad Y_1 = x_2, \quad Y_2 = x_4, \quad Y_3 = x_4 x_1 - x_2 x_3.$$

Therefore, in view of (6.9), (6.11) we have $J_1 = J_1(Y_1, Y_2, Y_3)$ and $\frac{\partial J_1}{\partial Y_3} \neq 0$.

Consequently, as $\delta_2 Y_1 = \delta_2 Y_2 = 0$, one gets

$$\delta_2 J_1 = \frac{\partial J_1}{\partial Y_1} \delta_2 Y_1 + \frac{\partial J_1}{\partial Y_2} \delta_2 Y_2 + \frac{\partial J_1}{\partial Y_3} \delta_2 Y_3 = \frac{\partial J_1}{\partial Y_3} \delta_2 Y_3.$$

This implies

$$(6.12) \quad \delta_2 Y_3 = 0.$$

By substituting in (6.12) the expression for Y_3 given by (6.11) we arrive to

$$(6.13) \quad b_3 = d_1 = 0, \quad b_1 = d_3 = \rho,$$

for some $\rho \in \mathbb{C}$.

We now use the equation $\delta_1 J = 0$ which can be written as

$$(6.14) \quad \delta_1 J = \frac{\partial J}{\partial Y_1} \delta_1 Y_1 + \frac{\partial J}{\partial Y_2} \delta_1 Y_2 + \frac{\partial J}{\partial Y_3} \delta_1 Y_3 = 0,$$

One can show that

$$\begin{aligned} \delta_1 Y_1 &= \rho Y_1, \\ \delta_1 Y_2 &= \rho Y_2, \\ \delta_1 Y_3 &= v_1 Y_1^2 + v_2 Y_2^2 + v_3 Y_1 Y_2, \end{aligned}$$

where $v_1 = d_2 - c_1$, $v_2 = a_3 - b_4$, $v_3 = a_1 - b_2 - c_3 + d_4$.

Hence, (6.14) yields

$$\rho Y_1 \frac{\partial J}{\partial Y_1} + \rho Y_2 \frac{\partial J}{\partial Y_2} + (v_1 Y_1^2 + v_2 Y_2^2 + v_3 Y_1 Y_2) \frac{\partial J}{\partial Y_3} = 0.$$

This equation possesses two rational functionally independent solutions $J_1(Y_1, Y_2, Y_3)$, $J_2(Y_1, Y_2, Y_3)$ only if

$$\rho = 0, \quad v_1 = v_2 = v_3 = 0.$$

which gives

$$(6.15) \quad a_1 = \epsilon_1 + b_2, \quad c_3 = \epsilon_1 + d_4, \quad c_1 = d_2 = \zeta_1, \quad a_3 = b_4 = \zeta_2, \quad \epsilon_1, \zeta_1, \zeta_2 \in \mathbb{C}.$$

After substitutions of (6.13), (6.15) in (6.2) the matrix R is written as

$$R = \begin{pmatrix} b_2 + \epsilon_1 & a_2 & \zeta_2 & a_4 \\ 0 & b_2 & 0 & \zeta_2 \\ \zeta_1 & c_2 & d_4 + \epsilon_1 & c_4 \\ 0 & \zeta_1 & 0 & d_4 \end{pmatrix}.$$

Now, consider the characteristic polynomial $P(\lambda)$ of R

$$P(\lambda) = \lambda^4 + P_1\lambda^3 + P_2\lambda^2 + P_3\lambda + P_4,$$

where

$$\begin{aligned} P_1 &= -2(b_2 + d_4 + \epsilon_1), \\ P_2 &= 3b_2\epsilon_1 - 2\zeta_1\zeta_2 + 3\epsilon_1d_4 + 4b_2d_4 + b_2^2 + \epsilon_1^2 + d_4^2, \\ P_3 &= -(d_4 + b_2 + \epsilon_1)(2b_2d_4 + b_2\epsilon_1 + d_4\epsilon_1 - 2\zeta_1\zeta_2), \\ P_4 &= (b_2d_4 - \zeta_1\zeta_2)(b_2d_4 + b_2\epsilon_1 + d_4\epsilon_1 - \zeta_1\zeta_2 + \epsilon_1^2). \end{aligned}$$

As above, in view of (6.1), (6.2) all eigenvalues of R must be equal to 0 and therefore $P_i = 0$, $1 \leq i \leq 4$. This system gives

$$\epsilon_1 = 0, \quad b_2 = \eta_1, \quad d_4 = -\eta_1, \quad \eta_1^2 + \zeta_1\zeta_2 = 0,$$

and the monodromy matrix T_2 becomes

$$T_2 = \begin{pmatrix} \eta_1 + 1 & a_2 & \zeta_2 & a_4 \\ 0 & \eta_1 + 1 & 0 & \zeta_2 \\ \zeta_1 & c_2 & 1 - \eta_1 & c_4 \\ 0 & \zeta_1 & 0 & 1 - \eta_1 \end{pmatrix}.$$

The matrix T_1T_2 has the eigenvalues $\{1, 1, 1, 1\}$ which contradicts to c) of Lemma 5.1 and proves our claim. \square

Due to our definition of integrability we deduce from Theorem 6.2 the following

Theorem 6.3 *The planar three-body problem is meromorphically non-integrable near the Lagrangian parabolic solution.*

7. Final remarks

In the end of 19th century Poincaré [18] indicated some qualitative phenomena in the behavior of phase trajectories which prevent the appearance of new integrals of a Hamiltonian system besides those which are present, but fail to form a set sufficient for complete integrability.

Let M^{2n} be the phase space, and $H : M^{2n} \rightarrow \mathbb{R}$, $H = H_0 + \epsilon H_1 + O(\epsilon^2)$ the Hamiltonian function. Suppose that for $\epsilon = 0$ the corresponding Hamiltonian system has an m -dimensional hyperbolic invariant torus T_0^m . According to the Graff's theorem [5], for small ϵ the perturbed system has an invariant hyperbolic torus T_ϵ^m depending analytically on ϵ . It can be shown that T_ϵ^m has asymptotic invariant manifolds Λ^+ and Λ^- filled with trajectories which tend to the torus T_ϵ^m as $t \rightarrow +\infty$ and $t \rightarrow -\infty$ respectively. In integrable Hamiltonian systems such manifolds (called also *separatrices*), as a rule, coincide. In the nonintegrable cases, the situation is different: asymptotic surfaces can have transverse intersection forming a complicated tangle which prevent the appearance of new integrals. For a modern presentation of these results see, for example, [7].

The method of splitting of asymptotic surfaces was applied to the three-body problem by many authors. In his book [13] J.K. Moser described a technique which use the symbolic dynamics associated with a transverse homoclinic point. Applying this method, it was shown in [9] that under certain assumptions the planar circular restricted three-body problem does not possess an additional real analytic integral. The similar result for the Sitnikov problem and the collinear three-body problem can be found in [13], [10]. The existence and the transverse intersection of stable and unstable manifolds along some periodic orbits in the planar three-body problem where two masses are sufficiently small was established in [11], using the results obtained in [12].

It is necessary to note that Theorem 6.3 implies the nonexistence of a complete set of complex analytic first integrals for the general planar three-body problem. To prove the nonexistence of real analytic integrals one should use some heteroclinic phenomena and can propose the following line of reasoning : Let M^∞ be the infinity manifold, then the taken Lagrangian parabolic orbit is biasymptotic to it. This is a weakly hyperbolic invariant manifold and the reference orbit is a heteroclinic orbit to different periodic orbits sitting in M^∞ . The dynamical interpretation of Theorem 6.3 seems to be the transversality of the invariant stable and

unstable manifolds of M^∞ , along this orbit. A combination of passages near several of these orbits (there is all the family obtained by rotation) should allow to prove the existence of a heteroclinic chain. This, in turn, gives rise to an embedding of a suitable subshift, with lack of predictability, chaos and implies the nonexistence of real analytic integrals.

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Appendix A. The matrices A_∞ , A , B , R , J

$$A_\infty = A_{\infty,ij}, 1 \leq i, j \leq 4.$$

$$\begin{aligned} A_{\infty,11} &= \frac{1}{4} \frac{12\alpha + 5\beta + 5\beta\alpha^2 + 26\alpha\beta + 12\alpha^2}{\alpha S_1}, & A_{\infty,12} &= \frac{3}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, \\ A_{\infty,13} &= -2 \frac{S_1}{S_2^2 S_3^3 \alpha^4 \beta}, & A_{\infty,14} &= 0, \\ A_{\infty,21} &= \frac{3}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, & A_{\infty,22} &= -\frac{1}{4} \frac{-12\alpha + \beta + \beta\alpha^2 - 2\alpha\beta - 12\alpha^2}{\alpha S_1}, \\ A_{\infty,23} &= 0, & A_{\infty,24} &= -2 \frac{S_1}{S_2^2 S_3^3 \alpha^4 \beta}, \\ A_{\infty,31} &= \frac{1}{8} \frac{\alpha^2 \beta S_3^3 (\alpha+1) S_2^3 (2\alpha + 13\beta + 13\beta\alpha^2 + 24\alpha\beta + 2\alpha^2)}{S_1^3}, \\ A_{\infty,32} &= \frac{3}{8} \frac{\sqrt{3}(\beta + 2\alpha + 4\alpha\beta + \beta\alpha^2 + 2\alpha^2)(\alpha-1)\beta\alpha^2 S_3^3 S_2^3}{S_1^3}, \\ A_{\infty,33} &= -\frac{1}{4} \frac{\beta(5\alpha^2 + 14\alpha + 5)}{\alpha S_1}, & A_{\infty,34} &= -\frac{3}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, \\ A_{\infty,41} &= \frac{3}{8} \frac{\sqrt{3}(\beta + 2\alpha + 4\alpha\beta + \beta\alpha^2 + 2\alpha^2)(\alpha-1)\beta\alpha^2 S_3^3 S_2^3}{S_1^3}, \\ A_{\infty,42} &= \frac{1}{8} \frac{\alpha^2 \beta S_3^3 (\alpha+1) S_2^3 (-10\alpha + 7\beta + 7\beta\alpha^2 - 12\alpha\beta - 10\alpha^2)}{S_1^3}, \\ A_{\infty,43} &= -\frac{3}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, & A_{\infty,44} &= \frac{1}{4} \frac{\beta(10\alpha + \alpha^2 + 1)}{\alpha S_1}. \end{aligned}$$

$$A = (A_{ij}), 1 \leq i, j \leq 4.$$

$$\begin{aligned} A_{11} &= -\frac{1}{4} \frac{(\alpha+1)(\alpha\beta + 4\alpha + \beta)}{\alpha S_1}, & A_{12} &= \frac{1}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, \\ A_{13} &= 2 \frac{S_1}{S_2^2 S_3^3 \alpha^4 \beta}, & A_{14} &= 0, & A_{21} &= \frac{1}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, \\ A_{22} &= -\frac{1}{4} \frac{10\alpha\beta + 3\beta\alpha^2 + 3\beta + 4\alpha^2 + 4\alpha}{\alpha S_1}, & A_{24} &= 2 \frac{S_1}{S_2^2 S_3^3 \alpha^4 \beta}, \\ A_{31} &= -\frac{1}{8} \frac{\alpha^2 \beta^2 S_3^3 (\alpha+1)(\alpha-1)^2 S_2^3}{S_1^3}, & A_{34} &= -\frac{1}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, \\ A_{23} &= 0, & A_{32} &= \frac{1}{8} \frac{\sqrt{3}(\alpha-1)(\alpha+1)^2 \alpha^2 \beta^2 S_2^3 S_3^3}{S_1^3}, & A_{33} &= \frac{1}{4} \frac{(\alpha-1)^2 \beta}{\alpha S_1}, \\ A_{41} &= \frac{1}{8} \frac{\sqrt{3}(\alpha-1)(\alpha+1)^2 \alpha^2 \beta^2 S_2^3 S_3^3}{S_1^3}, & A_{42} &= -\frac{3}{8} \frac{\alpha^2 \beta^2 S_3^3 (\alpha+1)^3 S_2^3}{S_1^3}, \\ A_{43} &= -\frac{1}{4} \frac{\sqrt{3}(\alpha+1)\beta(\alpha-1)}{\alpha S_1}, & A_{44} &= \frac{3}{4} \frac{(\alpha+1)^2 \beta}{\alpha S_1}. \end{aligned}$$

$$R = (R_{ij}), 1 \leq i, j \leq 4.$$

$$\begin{aligned} R_{11} &= -\frac{1}{2} \frac{2\alpha + \beta + 6\alpha\beta + 2\alpha^2 + \beta\alpha^2}{\alpha S_1}, & R_{12} &= -\frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, \\ R_{13} &= 0, & R_{14} &= 0, \\ R_{21} &= -\frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, & R_{22} &= \frac{1}{2} \frac{(\alpha + 1)(-2\alpha + \alpha\beta + \beta)}{\alpha S_1}, \\ R_{23} &= 0, & R_{24} &= 0, \\ R_{31} &= -\frac{1}{8} \frac{\beta\alpha^2 S_2^3 S_3^3 (\alpha + 1)(\alpha^2 + 6\beta\alpha^2 + \alpha + 13\alpha\beta + 6\beta)}{S_1^3}, \\ R_{32} &= -\frac{1}{8} \frac{\sqrt{3}(3\alpha^2 + 2\beta\alpha^2 + 7\alpha\beta + 3\alpha + 2\beta)(\alpha - 1)\beta\alpha^2 S_3^3 S_2^3}{S_1^3}, \\ R_{33} &= \frac{1}{2} \frac{(\alpha + \sqrt{3} + 2)(\alpha + 2 - \sqrt{3})\beta}{\alpha S_1}, & R_{34} &= \frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, \\ R_{41} &= -\frac{1}{8} \frac{\sqrt{3}(3\alpha^2 + 2\beta\alpha^2 + 7\alpha\beta + 3\alpha + 2\beta)(\alpha - 1)\beta\alpha^2 S_3^3 S_2^3}{S_1^3}, \\ R_{42} &= -\frac{1}{8} \frac{\beta\alpha^2 S_2^3 S_3^3 (\alpha + 1)(-5\alpha^2 + 2\beta\alpha^2 - 5\alpha - 9\alpha\beta + 2\beta)}{S_1^3}, \\ R_{43} &= \frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, & R_{44} &= -\frac{1}{2} \frac{(\alpha + \sqrt{3} + 2)(\alpha + 2 - \sqrt{3})\beta}{\alpha S_1}. \end{aligned}$$

$$J = (J_{ij}), 1 \leq i, j \leq 4.$$

$$\begin{aligned} J_{11} &= -\frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, & J_{12} &= \frac{1}{2} \frac{(\alpha + 1)(-2\alpha + \alpha\beta + \beta)}{\alpha S_1}, \\ J_{13} &= 0, & J_{14} &= 0, \\ J_{21} &= \frac{1}{2} \frac{2\alpha + \beta + 6\alpha\beta + 2\alpha^2 + \beta\alpha^2}{\alpha S_1}, & J_{22} &= \frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, \\ J_{23} &= 0, & J_{24} &= 0, & J_{31} &= -\frac{1}{4} \frac{\sqrt{3}(\alpha - 1)(\alpha + 1)^2 \alpha^2 \beta^2 S_2^3 S_3^3}{S_1^3}, \\ J_{32} &= \frac{1}{4} \frac{\beta^2 \alpha^2 S_2^3 S_3^3 (\alpha + 1)(\alpha^2 + 4\alpha + 1)}{S_1^3}, \\ J_{33} &= \frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}, & J_{34} &= -\frac{1}{2} \frac{2\alpha + \beta + 6\alpha\beta + 2\alpha^2 + \beta\alpha^2}{\alpha S_1}, \\ J_{41} &= \frac{1}{4} \frac{\beta^2 \alpha^2 S_2^3 S_3^3 (\alpha + 1)(\alpha^2 + 4\alpha + 1)}{S_1^3}, \\ J_{42} &= \frac{1}{4} \frac{\sqrt{3}(\alpha - 1)(\alpha + 1)^2 \alpha^2 \beta^2 S_2^3 S_3^3}{S_1^3}, \\ J_{43} &= -\frac{1}{2} \frac{(\alpha + 1)(-2\alpha + \alpha\beta + \beta)}{\alpha S_1}, & J_{44} &= -\frac{1}{2} \frac{\sqrt{3}(\alpha + 1)\beta(\alpha - 1)}{\alpha S_1}. \end{aligned}$$